J. Math. Anal. Appl. 393 (2012) 671–679

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

The Nehari manifold and the existence of multiple solutions for a singular quasilinear elliptic equation

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ARTICLE INFO

Article history:
Received 11 May 2011
Available online 12 April 2012
Submitted by P.J. McKenna

Keywords:
Variational methods
Nehari manifold and fibrering maps
Singular quasilinear elliptic equation
Concave and convex nonlinearities

ABSTRACT

In this paper, we are concerned with the existence of multiple positive solutions for the singular quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda h(x)|u|^{m-2}u + H(x)|u|^{n-2}u, & x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $1 < p < N$, $0 \leq a < (N-p)/p$, $1 < m < p < n < pN/(N-(1+a)p)$, $\lambda > 0$. $h(x), H(x)$ are Lebesgue measurable functions which may change sign on Ω. We prove that there exist at least two positive solutions by using the Nehari manifold and the fibrering maps associated with the energy functional for this problem.

1. Introduction and main results

Recently, by a critical point result due to Bonanno [1] and Ricceri [2], Kristály and Varga [3] have established the existence of multiple solutions for the singular elliptic problem

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) = \lambda|x|^{-2b}f(u) + \mu|x|^{-2c}g(u), & x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $0 < a < (N-2)/2$, $a \leq b, c < a+1$, $f(u)$ is sublinear at infinity and superlinear at the origin, and g satisfies $|g(u)| \leq C_1(1+|u|^{q-1})$ with $2 < q < \min\{2N/(N-2), 2(N-2c)/(N-2(a+1))\}$. The parameter $\lambda \in (\delta_1, \delta_2) \subset (0, \infty)$, $\mu \in (0, \delta_3)$ with some $\delta_i > 0$ for $i = 1, 2, 3$.

It is worth noticing that problem (1.1) may be viewed in particular as a singular elliptic problem involving concave–convex nonlinearities.

Furthermore, Assuncão et al. in [4] considered the existence of nontrivial solutions for the singular quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = g(x,u) + |x|^{-bq}|u|^{q-2}u, & x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.2}$$

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doi:10.1016/j.jmaa.2012.03.015

with $0 \le a < (N-p)/p$, $a < b < a+1$, $d = a+1-b$, $q = pN/(N-pd)$ is the critical Sobolev–Hardy exponent and $g(x,s)$ can change sign and has subcritical growth at infinity, that is,

$$\lim_{|s|\to\infty} \frac{g(x,s)}{|x|^{-bq}|s|^{q-1}} = 0 \tag{1.3}$$

uniformly with respect to $x \in \Omega$. In particular, if $N - p(a+1) > (p-1)c$ and $g(x,s) = \lambda |x|^{c-p(a+1)}|u|^{p-2}u$, then there exists $\delta_1 > 0$ such that $\lambda \in (\lambda_1 - \delta_1, \lambda_1)$, problem (1.2) has at least two pairs of nontrivial solutions, where λ_1 is the first eigenvalue of the operator $-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$ relative to the homogeneous Dirichlet problem in Ω.

Similar consideration can be found in [5–10] and the references therein. We note that the lower and upper-solutions method is used to obtain the existence results in [6,8].

For $a = 0$ and $p = 2$, by using the Nehari manifold and the fibrering maps method, Brown [7] and Wu [9] proved the existence of at least two positive solutions for a semilinear elliptic problem

$$\begin{cases} -\Delta u(x) = \lambda h(x) u^q + H(x) u^p, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < q < 1 < p < (N+2)/(N-2)$, $\lambda > 0$, and the functions $h(x), H(x)$ are sign-changing on Ω.

In this paper, we are motivated from [3,7,9] and concerned with the existence and multiplicity results of positive solutions to the singular quasilinear elliptic problem with concave–convex nonlinearities

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda h(x)|u|^{m-2}u + H(x)|u|^{n-2}u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a bounded domain containing the origin with smooth boundary $\partial\Omega$, and $1 < p < N$, $0 \le a < (N-p)/p$, $1 < m < p < n < Np/(N-(1+a)p)$ and $\lambda > 0$ is a parameter, and $h(x), H(x) : \Omega \to \mathbb{R}^1$ are Lebesgue measurable functions which are somewhere positive but may change sign on Ω.

For $a \ne 0$ and $p \ne 2$, it seems like that there is little information on the existence of multiple positive solutions of problem (1.5) by using the Nehari manifold and the fibrering maps method.

The main goal of this paper is to generalize a result in [3,7,9] to a broader class of singular quasilinear elliptic equation like in (1.5).

In order to state our main results, we introduce some weighted spaces. If $\alpha \in \mathbb{R}^1$ and $r \ge 1$, we define $L^r(\Omega, |x|^{-\alpha})$ as being the subspace of $L^r(\Omega)$, of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}^1$, satisfying

$$\|u\|_{r,\alpha} = \|u\|_{L^r(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^r dx\right)^{1/r} < \infty. \tag{1.6}$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W^{1,p}(\Omega, |x|^{-ap})$ (respectively $W_0^{1,p}(\Omega, |x|^{-ap})$) as being the closure of $C^\infty(\Omega)$ (respectively $C_0^\infty(\Omega)$) with respect to the norm

$$\|u\| = \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}. \tag{1.7}$$

The following Sobolev–Hardy inequality with weights was proved by Caffarelli et al. in [11]. There is a constant $K_{a,b} > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\right)^{p/p^*} \le K_{a,b} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \tag{1.8}$$

where $-\infty < a < (N-p)/p$, $a \le b < a+1$, $d = a+1-b$ and $p^* = pN/(N-pd)$.

From the boundedness of Ω and the standard approximation arguments, it is easy to see that (1.8) holds on $W_0^{1,p}(\Omega, |x|^{-ap})$.

The following compact embedding theorem is an extension of the classical Rellich–Kondrachov compactness theorem; see [12].

Lemma 1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $N \ge 3$, $-\infty < a < (N-p)/p$. Then the embedding $X \equiv W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact if $1 \le r < pN/(N-p)$ and $\alpha < (1+a)r + N(1-r/p)$.*

For every $1 \le r \le pN/(N-p)$ and $\alpha \le (1+a)r + N(1-r/p)$, we denote

$$S_{\alpha,r} = \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\left(\int_\Omega |x|^{-\alpha}|u|^r dx\right)^{p/r}}, \tag{1.9}$$

where $S_{\alpha,r} > 0$ is the best Sobolev constant of the embedding $X \hookrightarrow L^r(\Omega, |x|^{-\alpha})$.

In this paper, we make the following assumptions.

(A_1) $1 < p < N$, $0 \le a < (N-p)/p$, $1 < m < p < n < Np/(N-(1+a)p)$ and $\lambda > 0$.
(A_2) $h(x)$ is Lebesgue measurable and sign-changing in Ω, and positive on a non-empty open subset of Ω. Further, there exists $\alpha \le (1+a)m + N(1-m/p)$ such that $h(x)|x|^{\alpha} \in L^{\infty}(\Omega)$.
(A_3) $H(x)$ is Lebesgue measurable and sign-changing in Ω, and positive on a non-empty open subset of Ω. Further, there exists $\gamma \le (1+a)n + N(1-n/p)$ such that $H(x)|x|^{\gamma} \in L^{\infty}(\Omega)$.
(A_4) The parameters m, n, p satisfy

$$(n-m)^{n-m}m^{n-p} \le (p-m)^{p-m}(pn)^{n-p}.$$

Definition 1. We say that $u \in X \equiv W_0^{1,p}(\Omega; |x|^{-ap})$ is a weak solution of problem (1.5) if

$$\int_{\Omega} |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx - \lambda \int_{\Omega} h(x)|u|^{m-2}u\varphi dx - \int_{\Omega} H(x)|u|^{n-2}u\varphi dx = 0 \tag{1.10}$$

for all $\varphi \in C_0^{\infty}(\Omega)$.

Obviously, the solutions of problem (1.5) correspond to critical points of the energy functional

$$J_{\lambda}(u) = \frac{1}{p}\int_{\Omega} |x|^{-ap}|\nabla u|^p dx - \frac{\lambda}{m}\int_{\Omega} h(x)|u|^m dx - \frac{1}{n}\int_{\Omega} H(x)|u|^n dx \tag{1.11}$$

with $u \in X$.

From (1.9) and assumptions (A_1)–(A_3), it follows that $J_{\lambda}(u)$ is well-defined in X, and $J_{\lambda}(u) \in C^1$. But, it follows from (A_1)–(A_3) that $J_{\lambda}(u)$ is not bounded below on the whole space X. In order to obtain existence results, we introduce the Nehari manifold

$$S_{\lambda}(\Omega) = \{u \in X : \langle J'_{\lambda}(u), u\rangle = 0\}, \tag{1.12}$$

where $\langle , \rangle$ denotes the usual duality between X and X^*. Thus $u \in S_{\lambda}(\Omega)$ if and only if

$$\int_{\Omega} |x|^{-ap}|\nabla u|^p dx - \lambda\int_{\Omega} h(x)|u|^m dx - \int_{\Omega} H(x)|u|^n dx = 0. \tag{1.13}$$

On the Nehari manifold $S_{\lambda}(\Omega)$, we have

$$J_{\lambda}(u) = \left(\frac{1}{p} - \frac{1}{m}\right)\int_{\Omega} |x|^{-ap}|\nabla u|^p dx + \left(\frac{1}{m} - \frac{1}{n}\right)\int_{\Omega} H(x)|u|^n dx \tag{1.14}$$

$$= \left(\frac{1}{p} - \frac{1}{n}\right)\int_{\Omega} |x|^{-ap}|\nabla u|^p dx - \lambda\left(\frac{1}{m} - \frac{1}{n}\right)\int_{\Omega} h(x)|u|^m dx \tag{1.15}$$

$$= \lambda\left(\frac{1}{p} - \frac{1}{m}\right)\int_{\Omega} h(x)|u|^m dx + \left(\frac{1}{p} - \frac{1}{n}\right)\int_{\Omega} H(x)|u|^n dx. \tag{1.16}$$

It is useful to understand $S_{\lambda}(\Omega)$ in terms of the fibrering maps $\phi_u(t) = J_{\lambda}(tu)(t > 0)$. It is clear that, if u is a local minimizer of J_{λ}, then ϕ_u has a local minimum at $t = 1$.

In this paper, we will use the Nehari manifold and the fibrering maps method to establish the multiple solutions of (1.5). Our main result is the following.

Theorem 1. *Suppose that assumptions* (A_1)–(A_4) *are fulfilled. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$, *problem* (1.5) *has at least two positive solutions.*

Remark 1. We take $p = 2$, $h(x) = |x|^{-2b}$, $H(x) = |x|^{-2c}$. Then assumption (A_2) implies

$$2b \le \alpha \le (1+a)m + N(1-m/2). \tag{1.17}$$

Similarly, assumption (A_3) gives that

$$2c \le \gamma \le (1+a)n + N(1-n/2). \tag{1.18}$$

It is clear that

$$(1+a)m + N(1-m/2) > 2(a+1), \qquad (1+a)n + N(1-n/2) > 2(a+1). \tag{1.19}$$

But, the restrictions on b, c in [3] are $a \le b, c < a+1$. These mean that the parameters b, c in Theorem 1 are in a broader region than that in [3].

This paper is organized as follows. In Section 2, we derive the main properties of the Nehari manifold and the fibrering maps associated with (1.5). By the results in Section 2, we give the proof of the main theorem in Section 3.

2. Properties of the Nehari manifold and Fibrering maps

In this section, we give a fairly complete description of the fibrering maps $\phi_u(t) = J_\lambda(tu)$, which were first introduced by Drabek and Pohozaev in [13] and are also discussed by Brown and Wu in [14,15].

By the definition, we have for $u \in X$ that

$$\phi_u(t) \equiv J_\lambda(tu) = \frac{1}{p}t^p \int_\Omega |x|^{-ap}|\nabla u|^p dx - \frac{\lambda}{m}t^m \int_\Omega h(x)|u|^m dx - \frac{1}{n}t^n \int_\Omega H(x)|u|^n dx, \tag{2.1}$$

$$\phi_u'(t) = t^{p-1} \int_\Omega |x|^{-ap}|\nabla u|^p dx - \lambda t^{m-1} \int_\Omega h(x)|u|^m dx - t^{n-1} \int_\Omega H(x)|u|^n dx, \tag{2.2}$$

$$\phi_u''(t) = (p-1)t^{p-2} \int_\Omega |x|^{-ap}|\nabla u|^p dx - \lambda(m-1)t^{m-2} \int_\Omega h(x)|u|^m dx - (n-1)t^{n-2} \int_\Omega H(x)|u|^n dx. \tag{2.3}$$

It is easy to see that $u \in S_\lambda(\Omega)$ if and only if $\phi_u'(1) = 0$ and, more generally, that $\phi_u'(t) = 0$ if and only if $tu \in S_\lambda(\Omega)$, that is, elements in $S_\lambda(\Omega)$ correspond to stationary points of fibrering maps. Thus it is natural to divide $S_\lambda(\Omega)$ into three subsets $S_\lambda^+(\Omega)$, $S_\lambda^-(\Omega)$ and $S_\lambda^0(\Omega)$ corresponding to local minima, local maxima and points of inflection of fibrering maps and so we define

$$S_\lambda^+(\Omega) = \{u \in S_\lambda(\Omega) | \phi_u''(1) > 0\}, \tag{2.4}$$

$$S_\lambda^-(\Omega) = \{u \in S_\lambda(\Omega) | \phi_u''(1) < 0\}, \tag{2.5}$$

$$S_\lambda^0(\Omega) = \{u \in S_\lambda(\Omega) | \phi_u''(1) = 0\} \tag{2.6}$$

and note that if $u \in S_\lambda(\Omega)$, i.e., $\phi_u'(1) = 0$, then

$$\phi_u''(1) = (p-m) \int_\Omega |x|^{-ap}|\nabla u|^p dx - (n-m) \int_\Omega H(x)|u|^n dx \tag{2.7}$$

$$= (p-n) \int_\Omega |x|^{-ap}|\nabla u|^p dx + \lambda(n-m) \int_\Omega h(x)|u|^m dx. \tag{2.8}$$

We first establish the following results.

Lemma 2. *Suppose that u_0 is a local maximum or minimum for $J_\lambda(u)$ on $S_\lambda(\Omega)$. Then, if $u_0 \notin S_\lambda^0(\Omega)$, u_0 is a critical point of $J_\lambda(u)$.*

Proof. Denote

$$F(u) = \int_\Omega |x|^{-ap}|\nabla u|^p dx - \lambda \int_\Omega h(x)|u|^m dx - \int_\Omega H(x)|u|^n dx, \quad u \in X. \tag{2.9}$$

Consider the optimization problem

$$\min_{u \in S_\lambda(\Omega)} J_\lambda(u) \quad \text{subject to } F(u) = 0. \tag{2.10}$$

Hence, by the theory of Lagrange multiplier principle, there exists $\mu \in \mathbb{R}^1$ such that $J_\lambda'(u_0) = \mu F'(u_0)$. Thus,

$$\langle J_\lambda'(u_0), u_0 \rangle = \mu \langle F'(u_0), u_0 \rangle. \tag{2.11}$$

Since $u_0 \in S_\lambda(\Omega)$, we obtain

$$\langle J_\lambda'(u_0), u_0 \rangle = 0.$$

However,

$$\langle F'(u_0), u_0 \rangle = p \int_\Omega |x|^{-ap}|\nabla u_0|^p dx - \lambda m \int_\Omega h(x)|u_0|^m dx - n \int_\Omega H(x)|u_0|^n dx$$

$$= (p-m) \int_\Omega |x|^{-ap}|\nabla u_0|^p dx + (m-n) \int_\Omega H(x)|u_0|^n dx.$$

Hence, if $u_0 \notin S_\lambda^0(\Omega)$, $\langle F'(u_0), u_0 \rangle \neq 0$ and so by (2.11) $\mu = 0$ and $J_\lambda'(u_0) = 0$. Therefore, Lemma 2 is proved. □

Lemma 3. *Assume* (A_1)–(A_3). *Then $J_\lambda(u)$ is coercive and bounded below on $S_\lambda(\Omega)$.*

Proof. In fact, we have from (1.9) that

$$\left(\int_\Omega |x|^{-\alpha}|u|^r dx\right)^{p/r} \le S_{\alpha,r}^{-1}\int_\Omega |x|^{-ap}|\nabla u|^p dx \equiv S_{\alpha,r}^{-1}\|u\|^p$$

for $1 \le r \le pN/(N-p)$, $\alpha \le (1+a)r + N(1-r/p)$. Note that

$$\begin{aligned}\int_\Omega h(x)|u|^m dx &= \int_\Omega h(x)|x|^\alpha |x|^{-\alpha}|u|^m dx\\ &\le \|h|x|^\alpha\|_{L^\infty}\int_\Omega |x|^{-\alpha}|u|^m dx \le S_{\alpha,m}^{-m/p}h_\alpha\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{m/p},\end{aligned} \tag{2.12}$$

where $\alpha \le (1+a)m + N(1-m/p)$, $h_\alpha = \|h|x|^\alpha\|_\infty$. Thus

$$J_\lambda(u) \ge (p^{-1}-n^{-1})\|u\|^p - \lambda S_{\alpha,m}^{-m/p}(m^{-1}-n^{-1})h_\alpha\|u\|^m \ge \alpha_0\|u\|^p - C_1(\lambda h_\alpha)^{p/(p-m)}, \tag{2.13}$$

where $\alpha_0 = \frac{1}{2}(p^{-1}-n^{-1}) > 0$ and $C_1 > 0$ is independent of λ, h_α. So $J_\lambda(u)$ is coercive and bounded below on $S_\lambda(\Omega)$. □

We now investigate the fibrering maps $\phi_u(t)$ defined by (2.1). We will see that the essential nature of the maps is determined by the signs of $\int_\Omega h(x)|u|^m dx$ and $\int_\Omega H(x)|u|^n dx$. It is useful to consider the function

$$M_u(t) = t^{p-m}\int_\Omega |x|^{-ap}|\nabla u|^p dx - t^{n-m}\int_\Omega H(x)|u|^n dx. \tag{2.14}$$

Clearly, for $t > 0$, $tu \in S_\lambda(\Omega)$ if and only if t is a solution of

$$M_u(t) = \lambda\int_\Omega h(x)|u|^m dx. \tag{2.15}$$

Moreover, if $\int_\Omega H(x)|u|^n dx \le 0$, we get

$$M_u'(t) = (p-m)t^{p-m-1}\int_\Omega |x|^{-ap}|\nabla u|^p dx - (n-m)t^{n-m-1}\int_\Omega H(x)|u|^n dx \ge 0, \tag{2.16}$$

where $1 < m < p < n$. Then $M_u(t)$ is a strictly increasing function for $t \ge 0$.

If $\int_\Omega H(x)|u|^n dx > 0$, we have from $M_u'(t) = 0$ that there is a unique critical point t_0:

$$t_0 = \left(\frac{(p-m)\int_\Omega |x|^{-ap}|\nabla u|^p dx}{(n-m)\int_\Omega H(x)|u|^n dx}\right)^{1/(n-p)}. \tag{2.17}$$

Since $n > p > m$, we have $M_u(t) \to 0$ as $t \to 0^+$ and $M_u(t) \to -\infty$ as $t \to +\infty$.

Furthermore, the direction computation gives that

$$M_u''(t_0) = (n-m)(p-n)t_0^{n-m-2}\int_\Omega H(x)|u|^n dx < 0. \tag{2.18}$$

This shows that $M_u(t)$ is increasing in $(0, t_0)$ and decreasing for $t \ge t_0$. Thus t_0 is a single turning point if $\int_\Omega H(x)|u|^n dx > 0$.

Suppose $tu \in S_\lambda(\Omega)$. It follows from (2.7) and (2.16) that $\phi_{tu}''(1) = t^{m+1}M_u'(t)$ and if $M_u'(t) > 0$, then $tu \in S_\lambda^+(\Omega)$, and if $M_u'(t) < 0$, then $tu \in S_\lambda^-(\Omega)$.

In the following, we will now describe the nature of the fibrering maps $\phi_u(t)$ for all possible signs of $\int_\Omega H(x)|u|^n dx$ and $\int_\Omega h(x)|u|^m dx$.

If $\int_\Omega H(x)|u|^n dx \le 0$ and $\int_\Omega h(x)|u|^m dx \le 0$. Obviously, ϕ_u is an increasing function of t, thus no multiple of u lies in $S_\lambda(\Omega)$.

If $\int_\Omega H(x)|u|^n dx \le 0$ and $\int_\Omega h(x)|u|^m dx > 0$. It is clear that there is exactly one solution for (2.15), thus there exists $t = t(u)$ such that $\phi_u'(t) = 0$ and $t(u)u \in S_\lambda(\Omega)$. Obviously,

$$M_u'(t) = \lambda(p-m)t^{-1}\int_\Omega h(x)|u|^m dx + (p-n)t^{n-m-1}\int_\Omega H(x)|u|^n dx > 0.$$

So $t(u)u \in S_\lambda^+(\Omega)$. Thus the fibrering maps ϕ_u has a unique critical point at $t = t(u)$ which is a local minimum. Since $\lim_{t\to+\infty}\phi_u(t) = +\infty$, then $\phi_u(t)$ is initially decreasing and eventually increasing with a single turning point.

If $\int_\Omega H(x)|u|^n dx > 0$ and $\int_\Omega h(x)|u|^m dx \le 0$. Then $M_u(t)$ is initially increasing and eventually decreasing with a single turning point. It is clear that there is exactly one positive solution for (2.15). Thus there is again a unique value $t(u) > 0$ such that $t(u)u \in S_\lambda(\Omega)$ and clearly $M_u'(t) < 0$, so in this case $t(u)u \in S_\lambda^-(\Omega)$. Hence the fibrering maps $\phi_u(t)$ has a unique critical point which is a local maximum. Since $\lim_{t\to+\infty}\phi_u(t) = -\infty$, then $\phi_u(t)$ is initially increasing and eventually decreasing with a single turning point.

Finally we consider the case $\int_\Omega H(x)|u|^n dx > 0$ and $\int_\Omega h(x)|u|^m dx > 0$, where the situation is more complicated. As in the previous case $M_u(t)$ is initially increasing and eventually decreasing with a single turning point. If $\lambda > 0$ is sufficiently large, (2.15) has no solution and so $\phi_u(t)$ has no critical points, in this case $\phi_u(t)$ is a decreasing function. Hence no multiple of u lies in $S_\lambda(\Omega)$. If, on the other hand, $\lambda > 0$ is sufficiently small, there are exactly two solutions $t_1(u) < t_2(u)$ of (2.15) with $M_u'(t_1) > 0$ and $M_u'(t_2) < 0$. Thus there are exactly two multiples of $u \in S_\lambda(\Omega)$, that is, $t_1(u)u \in S_\lambda^+(\Omega)$ and $t_2(u)u \in S_\lambda^-(\Omega)$. It follows that $\phi_u(t)$ has exactly two critical points, a local minimum at $t = t_1(u)$ and a local maximum at $t = t_2(u)$. Moreover $\phi_u(t)$ is decreasing in $(0, t_1)$, increasing in (t_1, t_2) and decreasing in (t_2, ∞).

The following result shows that when λ is small $\phi_u(t_0) > 0$ for some $t_0 > 0$ and all $u \in X \setminus \{0\}$.

Lemma 4. *Let* (A_1)–(A_3) *hold. Then there exist* $\lambda_1 > 0$ *and* $t_0 > 0$ *such that* $\phi_u(t_0) > 0$ *for all* $u \in X \setminus \{0\}$ *and* $0 < \lambda < \lambda_1$.

Proof. It is easy to see that if $\int_\Omega H(x)|u|^n dx \le 0$, then $\phi_u(t) > 0$ for t sufficiently large. If $\int_\Omega H(x)|u|^n dx > 0$, we denote

$$\psi_u(t) = \frac{t^p}{p}\int_\Omega |x|^{-ap}|\nabla u|^p dx - \frac{t^n}{n}\int_\Omega H(x)|u|^n dx. \tag{2.19}$$

Then $\psi_u(0) = 0$ and $\psi_u(t) \to -\infty$ as $\to +\infty$. Further,

$$\psi_u'(t) = t^{p-1}\int_\Omega |x|^{-ap}|\nabla u|^p dx - t^{n-1}\int_\Omega H(x)|u|^n dx. \tag{2.20}$$

Then elementary calculus shows that ψ_u takes on a maximum value of

$$\psi_u(t_0) = \left(\frac{1}{p} - \frac{1}{n}\right)\left(\frac{\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^n}{\left(\int_\Omega H(x)|u|^n dx\right)^p}\right)^{1/(n-p)},$$

where

$$t_0 = \left(\frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\int_\Omega H(x)|u|^n dx}\right)^{1/(n-p)}. \tag{2.21}$$

From Lemma 1, we have

$$\begin{aligned}\int_\Omega H(x)|u|^n dx &= \int_\Omega H(x)|x|^\gamma |x|^{-\gamma}|u|^n dx \\ &\le H_\gamma \int_\Omega |x|^{-\gamma}|u|^n dx \le H_\gamma S_{\gamma,n}^{-n/p}\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{n/p},\end{aligned}$$

where $\gamma \le (1+a)n + N(1 - n/p)$, $H_\gamma = \|H|x|^\gamma\|_\infty$, and $S_{\gamma,n}$ denotes the best Sobolev constant of the embedding of $X \hookrightarrow L^n(\Omega, |x|^{-\gamma})$ defined by (1.9). Hence

$$\psi_u(t_0) \ge \left(\frac{1}{p} - \frac{1}{n}\right)(H_\gamma S_{\gamma,n}^{-n/p})^{p/(p-n)} \equiv \delta > 0 \tag{2.22}$$

where δ is independent of u. Similarly, we have

$$\begin{aligned}\frac{t_0^m}{m}\int_\Omega h(x)|u|^m dx &\le \frac{t_0^m}{m}\|h|x|^\alpha\|_\infty \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{m/p} S_{\alpha,m}^{-m/p} \\ &= \frac{h_\alpha}{m}\left(\frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\int_\Omega H(x)|u|^n dx}\right)^{\frac{m}{n-p}}\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{\frac{m}{p}} S_{\alpha,m}^{-m/p} \\ &\le C_0 h_\alpha\left[\psi_u(t_0)\right]^{\frac{m}{p}}\end{aligned} \tag{2.23}$$

where $C_0 > 0$ is independent of u, $h_\alpha = \|h|x|^\alpha\|_\infty$. Thus

$$\phi_u(t_0) \ge \psi_u(t_0) - \lambda C_0 \psi_u(t_0)^{m/p} \ge \psi_u(t_0)^{m/p}(\psi_u(t_0)^{(p-m)/p} - \lambda C_0) \tag{2.24}$$

and so, since $\psi_u(t_0) \ge \delta$ for all $u \in X \setminus \{0\}$, it follows that $\phi_u(t_0) > 0$ for all nonzero u provided $\lambda < C_0^{-1}\delta^{(p-m)/p} = \lambda_1$. This completes the proof. □

Remark 2. It follows from Lemma 4 that, when $0 < \lambda < \lambda_1$, $\int_\Omega H(x)|u|^n dx > 0$ and $\int_\Omega h(x)|u|^m dx > 0$, $\phi_u(t)$ must have exactly two critical points as discussed in the remarks preceding Lemma 4.

Lemma 5. *If $0 < \lambda < \lambda_1$, then there exists $\delta_1 > 0$ such that $J_\lambda(u) \geq \delta_1$, for all $u \in S_\lambda^-(\Omega)$.*

Proof. Note that $u \in S_\lambda^-(\Omega)$, then $\phi_u''(1) < 0$, and ϕ_u has a positive global maximum at $t = 1$ and $\int_\Omega H(x)|u|^n dx > 0$. Thus it follows from (2.24) that

$$J_\lambda(u) = \phi_u(1) \geq \phi_u(t_0) \geq \psi_u(t_0)^{m/p}(\psi_u(t_0)^{(p-m)/p} - \lambda C_0) \geq \delta^{m/p}(\delta^{(p-m)/p} - \lambda C_0) \tag{2.25}$$

and the left hand side is uniformly bounded away from 0 provided that $0 < \lambda < \lambda_1$. $\square$

Lemma 6. *Assume* (A_1)–(A_4) *and* $0 < \lambda < \lambda_1$. *Then, for every $u \in S_\lambda(\Omega)$, $u \not\equiv 0$, we have*

$$\int_\Omega |x|^{-ap}|\nabla u|^p dx - \lambda \int_\Omega h(x)|u|^m dx - \int_\Omega H(x)|u|^n dx \neq 0. \tag{2.26}$$

That is, $S_\lambda^0(\Omega) = \emptyset$.

Proof. Suppose that the result is false. Then there exists $u \in S_\lambda^0(\Omega) \subseteq S_\lambda(\Omega)$ such that

$$\int_\Omega |x|^{-ap}|\nabla u|^p dx - \lambda \int_\Omega h(x)|u|^m dx = \int_\Omega H(x)|u|^n dx. \tag{2.27}$$

If $\int_\Omega H|u|^n dx \leq 0$, then

$$\phi_u''(1) = (p-m)\int_\Omega |x|^{-ap}|\nabla u|^p + (m-n)\int_\Omega H(x)|u|^n dx > 0$$

and $u \notin S_\lambda^0(\Omega)$. Therefore, we assume that $\int_\Omega H(x)|u|^n dx > 0$. Then it follows from (2.7) and (2.27) that

$$\phi_u''(1) = \lambda(p-m)\int_\Omega h(x)|u|^m dx + (p-n)\int_\Omega H(x)|u|^n dx = 0. \tag{2.28}$$

and

$$-\lambda \int_\Omega h(x)|u|^m dx = \frac{p-n}{p-m}\int_\Omega H(x)|u|^n dx. \tag{2.29}$$

However, we obtain by (2.7) that

$$\phi_u''(1) = (p-m)\int_\Omega |x|^{-ap}|\nabla u|^p dx + (m-n)\int_\Omega H(x)|u|^n dx = 0 \tag{2.30}$$

and

$$\int_\Omega |x|^{-ap}|\nabla u|^p dx = \frac{n-m}{p-m}\int_\Omega H(x)|u|^n dx > 0. \tag{2.31}$$

We can derive from (2.29) and (2.31) that

$$\phi_u(t) = \left(\frac{n-m}{p(p-m)}t^p + \frac{p-n}{m(p-m)}t^m - \frac{1}{n}t^n\right)\int_\Omega H(x)|u|^n dx. \tag{2.32}$$

In particular, we have (2.21) and the assumption (A_4) that

$$\phi_u(t_0) = \frac{(n-p)t_0^p}{p-m}\left(\frac{n-m}{pn} - \frac{1}{m}\left(\frac{p-m}{n-m}\right)^{\frac{p-m}{n-p}}\right)\int_\Omega H(x)|u|^n dx \leq 0. \tag{2.33}$$

This is a contradiction by Lemma 4. Therefore $S_\lambda^0(\Omega) = \emptyset$. $\square$

3. Existence of positive solutions

In this section, we will give the proofs of the existence of two positive solutions, one in $S_\lambda^+(\Omega)$ and the other in $S_\lambda^-(\Omega)$. By Lemma 6, we have $S_\lambda(\Omega) = S_\lambda^-(\Omega) \cup S_\lambda^+(\Omega)$.

Lemma 7. *Let* (A_1)–(A_4) *hold. If* $0 < \lambda < \lambda_1$, *there exists a minimizer of* J_λ *on* $S_\lambda^+(\Omega)$.

Proof. By Lemma 3, we note that J_λ is bounded below on $S_\lambda(\Omega)$ and so on $S_\lambda^+(\Omega)$, there exists a minimizing sequence $\{u_k\} \subseteq S_\lambda^+(\Omega)$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in S_\lambda^+(\Omega)} J_\lambda(u).$$

Since $J_\lambda(u_k)$ is coercive, $\{u_k\}$ is bounded in X. By Lemma 1, we may assume, without loss of generality, that $u_k \rightharpoonup u_0$ in X and $u_k \to u_0$ in $L^q(\Omega, |x|^{-\alpha})$ for $1 < q < Np/(N-p(1+a))$, $\alpha < q(1+a)q + N(1-q/p)$.

If we choose $u \in X$ such that $\int_\Omega h(x)|u|^m dx > 0$, by the remarks preceding Lemma 4, there exists $t_1(u)$ such that $t_1(u)u \in S_\lambda^+(\Omega)$ and $J_\lambda(t_1(u)u) < 0$. Hence, $\inf_{u\in S_\lambda^+(\Omega)} J_\lambda(u) < 0$ and $J_\lambda(u_k) < 0$.

Further, it follows from (1.15) that

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{n}\right)\int_\Omega |x|^{-ap}|\nabla u_k|^p dx - \lambda\left(\frac{1}{m} - \frac{1}{n}\right)\int_\Omega h(x)|u_k|^m dx \tag{3.1}$$

and so

$$\lambda\left(\frac{1}{m} - \frac{1}{n}\right)\int_\Omega h(x)|u_k|^m dx = \left(\frac{1}{p} - \frac{1}{n}\right)\int_\Omega |x|^{-ap}|\nabla u_k|^p dx - J_\lambda(u_k). \tag{3.2}$$

Letting $k \to \infty$, we see that $\int_\Omega h(x)|u_0|^m dx > 0$.

Suppose $u_k \not\to u_0$ in X, we may obtain a contradiction by considering the fibrering map $\phi_{u_0}(t)$. Since $\int_\Omega h(x)|u_0|^m dx > 0$, then there exists $t_0 > 0$ such that $t_0 u_0 \in S_\lambda^+(\Omega)$ and $\phi_{u_0}(t)$ is decreasing on $(0, t_0)$ and $\phi'_{u_0}(t_0) = 0$. Since $u_k \not\to u_0$ in X, then

$$\int_\Omega |x|^{-ap}|\nabla u_0|^p dx < \liminf_{k\to\infty} \int_\Omega |x|^{-ap}|\nabla u_k|^p dx.$$

By (2.2), we have

$$\phi'_{u_k}(t) = t^{p-1}\int_\Omega |x|^{-ap}|\nabla u_k|^p dx - \lambda t^{m-1}\int_\Omega h(x)|u_k|^m dx - t^{n-1}\int_\Omega H(x)|u_k|^n dx \tag{3.3}$$

and

$$\phi'_{u_0}(t) = t^{p-1}\int_\Omega |x|^{-ap}|\nabla u_0|^p dx - \lambda t^{m-1}\int_\Omega h(x)|u_0|^m dx - t^{n-1}\int_\Omega H(x)|u_0|^n dx. \tag{3.4}$$

Thus, we obtain from $\phi'_{u_0}(t_0) = 0$ that

$$\begin{aligned}\phi'_{u_k}(t_0) &= t_0^{p-1}\int_\Omega |x|^{-ap}(|\nabla u_k|^p - |\nabla u_0|^p)dx - \lambda t_0^{m-1}\int_\Omega h(x)(|u_k|^m - |u_0|^m)dx \\ &\quad - t_0^{n-1}\int_\Omega H(x)(|u_k|^n - |u_0|^n)dx. \end{aligned} \tag{3.5}$$

Since $u_k \to u_0$ in $L^m(\Omega, |x|^{-\alpha})$ and in $L^n(\Omega, |x|^{-\gamma})$, we derive

$$\liminf_{k\to\infty} \phi'_{u_k}(t_0) \ge t_0^{p-1} \liminf_{k\to\infty} \int_\Omega |x|^{-ap}(|\nabla u_k|^p - |\nabla u_0|^p)dx > 0. \tag{3.6}$$

This shows that $\phi'_{u_k}(t_0) > 0$ for k sufficiently large. Since $\{u_k\} \subseteq S_\lambda^+(\Omega)$, i.e. $\phi''_{u_k}(1) > 0$, it is easy to see that $\phi'_{u_k}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_k}(1) = 0$ for all k. Hence we must have $t_0 > 1$. Since $t_0 u_0 \in S_\lambda^+(\Omega)$ and $\phi'_{u_0}(t_0) = 0$, $\phi_{u_0}(t_0)$ is a local minimum, and $\phi_{u_0}(t_0) < \phi_{u_0}(1)$. Thus

$$J_\lambda(t_0 u_0) < J_\lambda(u_0) < \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in S_\lambda^+(\Omega)} J_\lambda(u)$$

and this is a contradiction. Hence $u_k \to u_0$ in X and so

$$J_\lambda(u_0) = \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in S_\lambda^+(\Omega)} J_\lambda(u).$$

Therefore, u_0 is a minimizer for J_λ on $S_\lambda^+(\Omega)$. □

Lemma 8. *Assume* (A_1)–(A_4) *and* $0 < \lambda < \lambda_1$. *Then, there exists a minimizer of* J_λ *on* $S_\lambda^-(\Omega)$.

Proof. By Lemma 5, there is $\delta_1 > 0$ such that $J_\lambda(u) \geq \delta_1 > 0$ for all $u \in S_\lambda^-(\Omega)$ and so $\inf_{u \in S_\lambda^-(\Omega)} J_\lambda(u) \geq \delta_1$. Hence there exists a minimizing sequence $\{u_k\} \subseteq S_\lambda^-(\Omega)$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u \in S_\lambda^-(\Omega)} J_\lambda(u) > 0.$$

By Lemma 3, $J_\lambda(u_k)$ is coercive and $\{u_k\}$ is bounded in X, thus we may assume that $u_k \rightharpoonup v$ in X and $u_k \to v$ in $L^q(\Omega, |x|^{-\alpha})$ for $1 < q < Np/(N - p(1+a))$, $\alpha < (1+a)q + N((p-q)/p)$. By (1.14), we have

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{m}\right) \int_\Omega |x|^{-ap} |\nabla u_k|^p dx + \left(\frac{1}{m} - \frac{1}{n}\right) \int_\Omega H(x)|u_k|^n dx \tag{3.7}$$

and, since $\lim_{k\to\infty} J_\lambda(u_k) > 0$ and

$$\lim_{k\to\infty} \int_\Omega H(x)|u_k|^n dx = \int_\Omega H(x)|v|^n dx, \tag{3.8}$$

we must obtain $\int_\Omega H(x)|v|^n dx > 0$. Hence there exists $t_3 > 0$ such that $t_3 v \in S_\lambda^-(\Omega)$.

Suppose $u_k \not\to v$ in X. Then,

$$\int_\Omega |x|^{-ap} |\nabla v|^p dx < \liminf_{k\to\infty} \int_\Omega |x|^{-ap} |\nabla u_k|^p dx.$$

Since $u_k \in S_\lambda^-(\Omega)$ and $\phi_{u_k}(t) \to -\infty$ as $t \to +\infty$, $J_\lambda(u_k)$ is a unique critical point which is a maximum as discussed in the remarks preceding Lemma 4. So, $J_\lambda(u_k) \geq J(tu_k)$ for all $t \geq 0$ and

$$\begin{aligned} J_\lambda(t_3 v) &= \frac{1}{p} t_3^p \int_\Omega |x|^{-ap} |\nabla v|^p dx - \frac{\lambda}{m} t_3^m \int_\Omega h(x)|v|^m dx - \frac{1}{n} t_3^n \int_\Omega H(x)|v|^n dx \\ &< \lim_{k\to\infty} \left[\frac{1}{p} t_3^p \int_\Omega |x|^{-ap} |\nabla u_k|^p dx - \frac{\lambda}{m} t_3^m \int_\Omega h(x)|u_k|^m dx - \frac{1}{n} t_3^n \int_\Omega H(x)|u_k|^n dx\right] \\ &= \lim_{k\to\infty} J_\lambda(t_3 u_k) \leq \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u \in S_\lambda^-(\Omega)} J_\lambda(u), \end{aligned} \tag{3.9}$$

which is a contradiction. Hence $u_k \to v$ in X. As the proof of Lemma 7, we know that v is a minimizer for J_λ on $S_\lambda^-(\Omega)$. This completes the proof. □

Proof of Theorem 1. By Lemmas 7 and 8, there exists $u_1 \in S_\lambda^+(\Omega)$ and $u_2 \in S_\lambda^-(\Omega)$ such that $J_\lambda(u_1) = \inf_{u \in S_\lambda^+(\Omega)} J_\lambda(u)$ and $J_\lambda(u_2) = \inf_{u \in S_\lambda^-(\Omega)} J_\lambda(u)$. Moreover, $J_\lambda(u_1) = J_\lambda(|u_1|)$, $J_\lambda(u_2) = J_\lambda(|u_2|)$ and $|u_1| \in S_\lambda^+(\Omega)$, $|u_2| \in S_\lambda^-(\Omega)$, so we may assume $u_1, u_2 > 0$. By Lemma 2, u_1 and u_2 are critical points of $J_\lambda(u)$ on X; hence they are two weak solutions of (1.5). □

Acknowledgments

The authors wish to express their gratitude to the referees for useful comments and suggestions.

References

[1] G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Anal. 54 (2003) 651–665.
[2] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, Math. Comput. Modelling 32 (2000) 1485–1494.
[3] A. Kristály, C. Varga, Multiple solutions for a degenerate elliptic equation involving sublinear terms at infinity, J. Math. Anal. Appl. 352 (2009) 139–148.
[4] R.B. Assuncão, P.C. Carrião, O.H. Miyagaki, Subcritical perturbations of a singular quasilinear elliptic equation involving the critical Hardy–Sobolev exponent, Nonlinear Anal. 66 (2007) 1351–1464.
[5] C.O. Alves, A.E. Hamidi, Nehari manifold and existence of positive solutions to a class of quasilinear problems, Nonlinear Anal. 60 (2005) 611–624.
[6] F. Brock, L. Iturriaga, J. Sánchez, P. Ubilla, Existence of positive solutions for p-Laplacian problems with weights, Commun. Pure Appl. Anal. 5 (4) (2006) 941–952.
[7] K.J. Brown, T.F. Wu, A fibering map approach to a semilinear elliptic boundary value problem, Electron. J. Differential Equations 69 (2007) 1–9.
[8] O.H. Miyagaki, R.S. Rodrigues, On positive solutions for a class of singular quasilinear elliptic systems, J. Math. Anal. Appl. 334 (2007) 818–833.
[9] T.F. Wu, On semilinear elliptic equations involving concave–convex nonlinearities and sign-changing weight function, J. Math. Anal. Appl. 318 (2006) 253–270.
[10] B. Xuan, Multiple solutions to a Caffarelli–Kohn–Nirenberg type equations with asymptotically linear term, Rev. Colombiana Mat. 37 (2003) 65–79.
[11] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Compos. Math. 53 (1984) 437–477.
[12] B.J. Xuan, The solvability of quasilinear Brezis–Nirenberg-type problems with singular weights, Nonlinear Anal. 62 (2005) 703–725.
[13] P. Drabek, S.I. Pohozaev, Positive solutions for the p-Laplacian: application of the fibrering method, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 703–726.
[14] K.J. Brown, The Nehari manifold for a semilinear elliptic equation involving a sublinear term, Calc. Var. 22 (2005) 483–495.
[15] K.J. Brown, T.F. Wu, A semilinear elliptic system involving nonlinear boundary condidition and sign changing weight function, J. Math. Anal. Appl. 337 (2008) 1326–1336.